# Topology versus Borel structure for actions

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$$\Longrightarrow U \supseteq B_i B_i^{-1} \supseteq V_i V_i^{-1} \ni 1.$$

□

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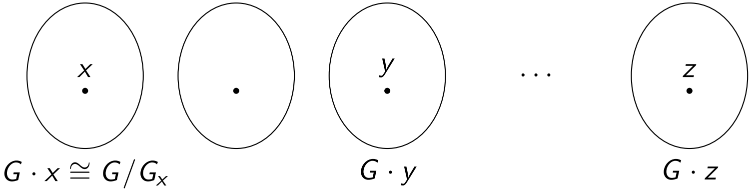

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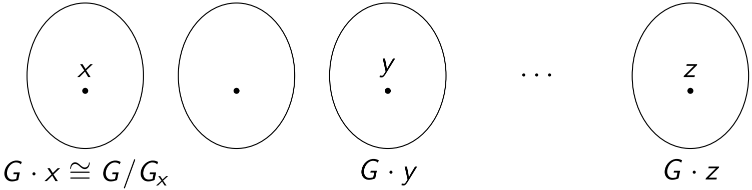


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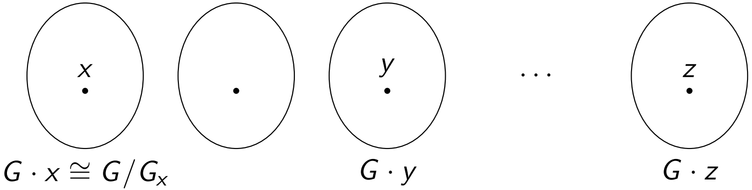


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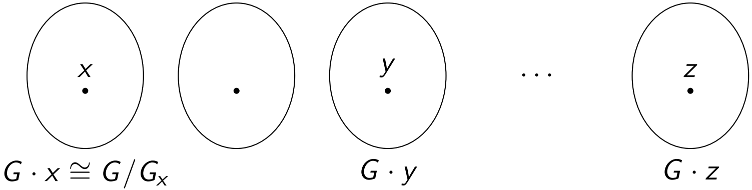


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**Corollary** *For a Polish $G$-sp $X$, topology may be refined to make orbwise open $B$ open.*
*(e.g., invariant)*

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3. Extend to various other contexts.
   - potentially open $n$-ary relations
   - non-Hausdorff (quasi-Polish) $G$-spaces
   - groupoid actions
   - actions preserving existing topology
   - non-second-countable actions (on point-free "spaces")

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- $\mathbf{\Pi}^0_2$ subspace of $\mathbb{S}^{\mathbb{N}}$, where $\mathbb{S} = \{0,1\}$ w/ $\{1\}$ open, and $\mathbf{\Pi}^0_2$ means $\bigcap_i (U_i \Rightarrow V_i)$
- continuous open $T_0$ quotient of $\mathbb{N}^{\mathbb{N}}$
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Fact *Quasi-Polish group $=$ Polish group.*

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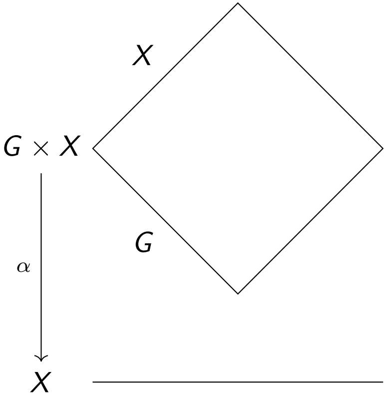

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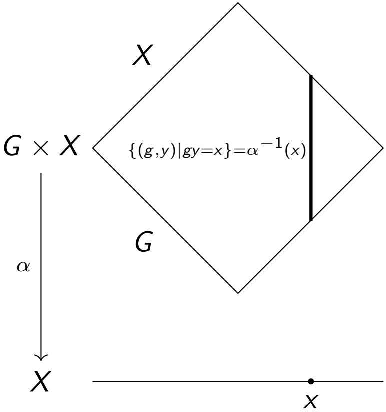

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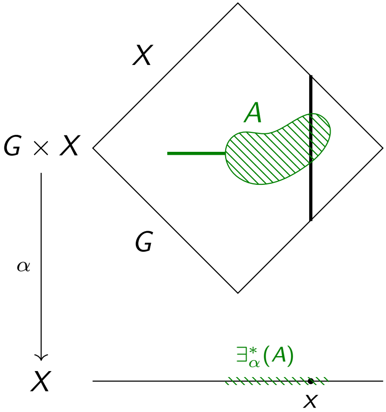


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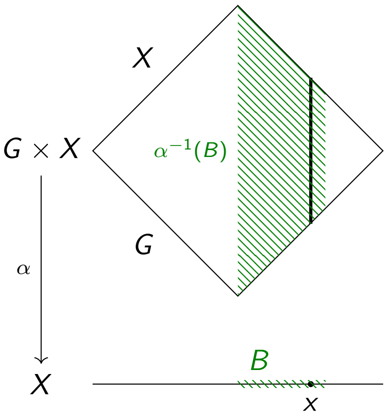


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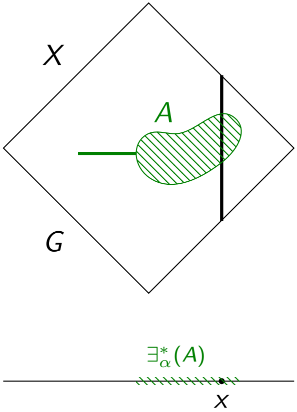


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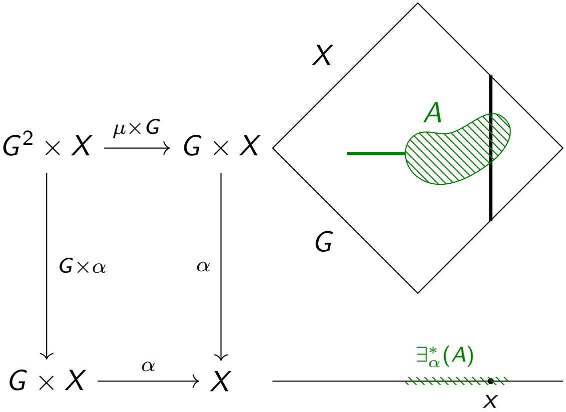


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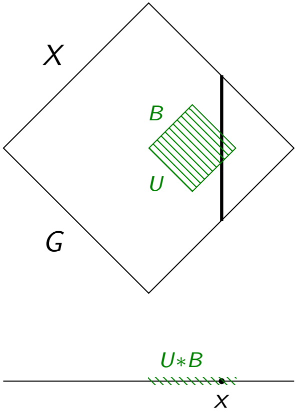


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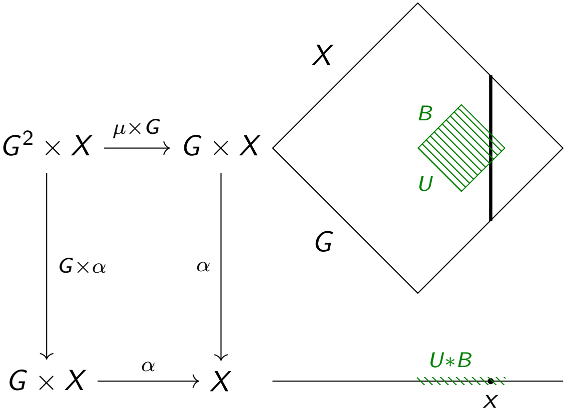


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**Core Theorem (C. 2022)**

*Let $G$ be a Polish group, $X$ be a quasi-Polish space with a Borel $G$-action s.t.*

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**Lemma** *Let $f : X \twoheadrightarrow Y$ be a Borel surj from a q-Pol sp to a st Borel sp. Suppose each fiber $f^{-1}(y)$ is equipped with a coarser q-Pol top "in a Borel way", and $f$ is cts wrt $\exists^*_f(\mathcal{O}(X))$. Then $Z :=$ smallest fbwise closed (in finer top) comgr (in coarser top) $\subseteq X$ is $\mathbf{\Pi}^0_2$, and $f|Z : Z \twoheadrightarrow Y$ is cts open $T_0$ quotient with $\exists^*_f = \exists^*_{f|Z}$.* □

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Proof. (i) $\Longrightarrow$ (ii),(iv)

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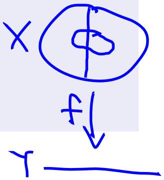


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**Theorem (Kunugui–Novikov)** *Let $f : X \to Y$ be a Borel map between st Borel spaces, $\mathcal{S} \subseteq \mathcal{B}(X)$ be ctble. If $A \in \mathcal{B}(X)$ is $f$-fiberwise a union of sets in $\mathcal{S}$, then*

$$A = \bigcup_{S \in \mathcal{S}} (f^{-1}(B_S) \cap S) \quad \text{for } B_S \in \mathcal{B}(Y).$$

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# Comparison with original proof

**Core Theorem (Becker–Kechris 1996)**

*Let $G$ be a Polish group, $X$ be a zero-dimensional Polish with a Borel $G$-action, $\mathcal{U} \subseteq \mathcal{O}(G)$ and $\mathcal{A} \subseteq \mathcal{O}(X)$ be countable bases s.t. $\mathcal{A}$ is a Boolean algebra and*

$$\mathcal{U} * \mathcal{A} \subseteq \mathcal{A}.$$

*Then $\langle \mathcal{U} * \mathcal{A} \rangle$ is a compat Polish top making action cts.*

The proof consists of showing:

1. the action is cts;
2. the topology is $T_3$;
3. the topology is strong Choquet.

Combining 1. and 2. with our Core Thm yields a Polish top realization.

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Corollary *For a quasi-Polish $G$-space $X$ and $B \in \mathbf{\Sigma}^0_\xi(X)$, $\xi \geq 2$, $U * B$ is open in a finer* **Polish** *topology $\subseteq \mathbf{\Sigma}^0_\xi(X)$ (0-d if $G$ is non-Archimedean).*

# Automatic continuity for actions

**Theorem (classical for Polish; C.)**

*Let $G$ be a Polish group, $X$ be a quasi-Polish $G$-space with a Borel action of $G$ via homeomorphisms. Then the action is jointly continuous.*

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# Groupoids

Definition A **groupoid** $G$ consists of two maps $G \overset{\sigma}{\underset{\tau}{\rightrightarrows}} G_0$ (src, tgt) and operations

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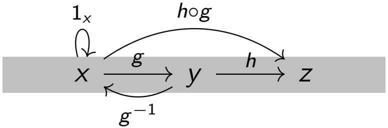

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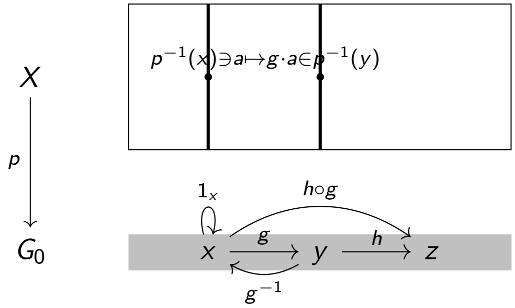


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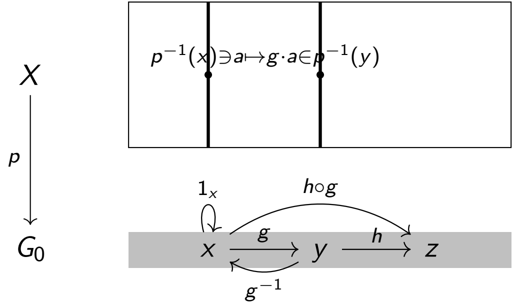


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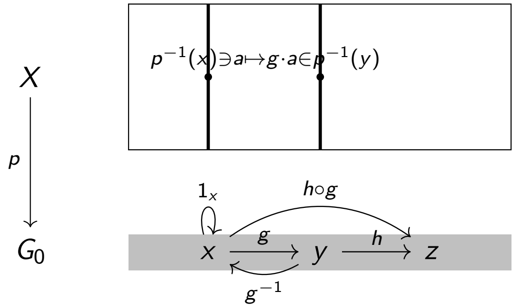


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Note Most open quasi-Polish groupoids are not Polish!

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*Let $G$ be an open q-Pol gpd, $p : X \to G_0$ a* **standard Borel bundle of q-Pol spaces** *with a $G$-action via homeos. Then $\exists$ global q-Pol top on $X$ restricting to fiberwise tops.*

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For a group(oid) action on $X$, we know $B \in \mathcal{B}(X)$ potentially open iff orbitwise open.
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Similarly for groupoids, multi-sorted structures, change of topology, . . .